Hutchcroft-Pete revisited (part one)

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Executive summary

Let Γ denote an infinite countable group

We will discuss *Kazhdan's Property (T)* for such groups

Our goal is to give a simpler proof of:

Theorem[Hutchcroft-Pete]
If Γ has Property (T), then it has a pmp action with cost one.

Random graphs on Γ

Let $\mathcal{G}$ denote a *random graph* with vertex set a subset of Γ

Formally, $\mathcal{G} : (\Omega, \mathbb{P}) \to \text{graphs}_\Gamma$ is a measurable function from some *auxiliary* probability space $(\Omega, \mathbb{P})$. Here $\text{graphs}_\Gamma \subset \{0,1\}^{\Gamma\times\Gamma}$ is the *space of graphs* on Γ

The *distribution* or *law* of $\mathcal{G}$ is the pushfoward measure $\mu = \mathcal{G}_*(\mathbb{P})$

We say that $\mathcal{G}$ is *invariant* if the random graphs $\gamma \cdot \mathcal{G}$ and $\mathcal{G}$ have the same distribution for all $\gamma \in \Gamma$. Equivalently, that $\Gamma \curvearrowright (\text{graphs}_\Gamma, \mu)$ is pmp

Concretely, the behaviour of $\mathcal{G}$ in any finite window doesn't depend on the location of the window

Cost via random graphs

If $\mathscr{G}$ is invariant then we define its *expected degree* as follows

$$\mathbb{E}\left[\deg_{\mathscr{G}}(0)\right] = \int_{\text{graphs}_\Gamma} \deg_G(0) d\mu(G),$$

where we denote by $0 \in \Gamma$ the identity element

Fact: The (infimal) *cost* of a group Γ is equal to $\inf_{\mathscr{G}} \frac{1}{2}\mathbb{E}\left[\deg_{\mathscr{G}}(0)\right]$, where $\mathscr{G}$ ranges over all *connected spanning* invariant random graphs on Γ

Intensity

The *intensity* of an invariant random graph $\mathscr{G}$

$$\mathrm{intensity}(\mathscr{G}) := \mathbb{P}\left[0 \in \mathscr{G}\right] = \mu(\{G \in \mathrm{graphs}_\Gamma \mid 0 \in G\})$$

It follows from Gaboriau's induction formula (or prove it directly) that

$$\mathrm{cost}(\Gamma) - 1 = \inf_{\mathscr{G}} \left\{ \frac{1}{2}\mathbb{E}\left[\deg_{\mathscr{G}}(0) \middle| 0 \in \mathscr{G}\right] - \mathrm{intensity}(\mathscr{G}) \right\},$$

where $\mathscr{G}$ ranges over all invariant random graphs *with a unique infinite component*.

The Hutchcroft-Pete argument

Recall that $\text{cost}(\Gamma) - 1 = \inf_{\mathcal{G}} \left\{ \frac{1}{2}\mathbb{E}\left[\deg_{\mathcal{G}}(0)\right] - \text{intensity}(\mathcal{G}) \right\}$

Let Γ be an infinite group with Kazhdan's Property (T), and fix a (finite) generating set $S \subset \Gamma$.

Then for any $p \in (0,1)$ there exists a random *subgraph* $\mathcal{G}$ of $\text{Cay}(\Gamma, S)$ with a unique infinite connected component and intensity p. In particular, such a group Γ has cost one.

Kazhdan's *Property (T)*

Let $\pi : \Gamma \to U(\mathscr{H})$ denote a *unitary representation* of Γ on a Hilbert space $\mathscr{H}$

A vector $\xi \in \mathscr{H}$ is *invariant* if $\pi(\gamma)\xi = \xi$ for all $\gamma \in \Gamma$

The vector is (Q, ε) *almost invariant* if $\|\pi(\gamma)\xi - \xi\| < \varepsilon\|\xi\|$ for all $\gamma \in Q$

A representation has *almost invariant vectors* if it admits (Q, ε) almost invariant vectors for all finite $Q \subseteq \Gamma$ and $\varepsilon > 0$

Γ has *Property (T)* if any representation with almost invariant vectors admits a nontrivial invariant vector

Can also define for locally compact second countable (lcsc) groups

Historical comments

If Γ has Property (T), then Γ must be finitely generated. Originally introduced to show that *any* lattice in a higher rank Lie group is finitely generated.

Why (T)? Can introduce a topology on the set of representations, and the property is exactly saying that the Trivial representation is isolated in that space.

Basic examples

- Finite groups
- $SL_n(\mathbb{Z})$ for $n \geq 3$
- *Random* groups with overwhelming probability (parameters)
- Automorphism groups of (sufficiently high rank) free groups

Basic nonexamples

- $\mathbb{Z}$
- Any group which is *amenable* and has Property (T) must be finite. Use that amenability is equivalent to saying the regular representation $l^2(\Gamma)$ has almost invariant vectors
- In particular, any group Γ with Property (T) must have that its abelianisation $\Gamma/[\Gamma, \Gamma]$ is *finite*

Glasner-Weiss

Theorem
Consider the collection of invariant random subsets of a group Γ (that is, the set $\mathrm{Prob}(\{0,1\}^\Gamma)^\Gamma$ of Γ-invariant probability measures on $\{0,1\}^\Gamma$)

Let $\mathcal{E} \subset \mathrm{Prob}(\{0,1\}^\Gamma)^\Gamma$ denote the subset of ergodic measures

Then $\mathcal{E}$ is closed *if and only* if Γ has Property (T)

…but in what topology?

Weak convergence

We will be interested in weak convergence in the space κ^Γ, where $\kappa = \{0,1\}^{\mathbb{N}}$ denotes the Cantor space, but for simplicity I'll just talk about $X := \{0,1\}^\Gamma$

Definition *(bad)*
A sequence of probability measures μ_n on $\{0,1\}^\Gamma$ *weakly converges* to μ if

$$\lim_{n\to\infty} \int_X f(x) d\mu_n(x) = \int_X f(x) d\mu(x)$$

for all continuous functions $f : X \to \mathbb{R}$

Weak convergence, *redux*

A subset A of a topological space is *clopen* if and only if its indicator function 1_A is continuous

Definition *(better)*
A sequence of probability measures μ_n on $\{0,1\}^\Gamma$ *weakly converges* to μ if

$$\lim_{n\to\infty} \mu_n(\{\omega \in \{0,1\}^\Gamma \mid \omega|_F = \alpha\}) = \mu(\{\omega \in \{0,1\}^\Gamma \mid \omega|_F = \alpha\}),$$

for all $F \subset \Gamma$ finite and $\alpha : F \to \{0,1\}$.

That is, if *the sampling probability in any finite window converges*.

Glasner-Weiss (hard direction)

Suppose Γ does *not* have Property (T). By Connes-Weiss, Γ admits an ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ with a sequence $A_n \subseteq X$ of *asymptotically invariant* sets,

$$\mu(A_n) = \frac{1}{2}, \text{ and for all } \gamma \in \Gamma \text{ we have } \mu(A_n \triangle \gamma \cdot A_n) \to 0$$

Let $\Sigma^n : (X, \mu) \to \{0,1\}^\Gamma$ denote *symbolic dynamics* with respect to A_n

$$\Sigma^n_x = \{\gamma \in \Gamma \mid \gamma^{-1}x \in A_n\}$$

Then *as invariant random subsets* Σ^n weakly converges to $\frac{1}{2}(\delta_\emptyset + \delta_\Gamma)$, which is manifestly nonergodic. But each Σ^n *is* ergodic.

Glasner-Weiss (easy direction)

Take μ_n to be ergodic probability measures on $X = \{0,1\}^\Gamma$ weakly converging to *nonergodic* μ

Form the product $Y = (X^{\mathbb{N}}, \bigotimes_{\mathbb{N}} \mu_n)$ and choose a measure λ in its ergodic decomposition that projects via π_n to μ_n in each factor

From a nontrivial Γ-invariant element of $L^2_0(X,\mu)$, for any $\varepsilon > 0$ we can find a *continuous* element $\varphi \in L^2_0(X,\mu)$ with $\langle \varphi, \gamma \cdot \varphi \rangle < \varepsilon$ for all $\gamma \in \Gamma$

Let $\varphi_n = \varphi \circ \pi_n$, so $\int_Y |\varphi_n - \gamma \cdot \varphi_n|^2 \, d\lambda = \int_X |\varphi - \gamma \cdot \varphi|^2 \, d\mu_n \to \int_X |\varphi - \gamma \cdot \varphi|^2 \, d\mu < \varepsilon$

So $L^2_0(Y,\lambda)$ has almost invariant vectors, but no *invariant* vectors by ergodicity, so Γ cannot have Property (T)

On the next episode…

- Bernoulli extensions
- Kazhdan optimal partitions
- Actions with finitely many infinite clusters

Fin.

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